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# ALMOST COMPLEX STRUCTURES ON TENSOR BUNDLES

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## 1. Introduction

It is well known that the tangent bundle of a $C^\infty$ manifold $M$ admits an almost complex structure if $M$ admits an affine connection [1], [5] or an almost complex structure [7], [8]. The main purpose of this paper is to investigate a similar problem for tensor bundles $T_s^rM$. We prove that if a Riemannian manifold $M$ admits an almost complex structure then so does $T_s^rM$ provided $r + s$ is odd. If $r + s$ is even a further condition is required on $M$. The proofs depend on some generalizations of the notions of lifting vector fields and derivations on $M$, which were defined previously only for tangent bundles and cotangent bundles [4], [7], [8], [9], [10].

## 2. Notations and definitions

(i) $M$ is a $C^\infty$ paracompact manifold of finite dimension $n$.

(ii) $F(M)$ is the ring of real-valued $C^\infty$ functions on $M$.

(iii) For $r + s > 0$, $T_s^rM$ is the bundle over $M$ of tensors of type $(r, s)$, contravariant of order $r$ and covariant of order $s$. $\pi$ is the projection of $T_s^rM$ onto $M$. We write $T_0^rM = T^rM$, $T_s^0M = T_sM$.

(iv) $\mathscr{T}_s^r(M)$ is the module over $F(M)$ of $C^\infty$ tensor fields of type $(r, s)$. We write $\mathscr{T}_0^r(M) = \mathscr{T}^r(M)$, $\mathscr{T}_s^0(M) = \mathscr{T}_s(M)$, and $\mathscr{T}_0^0(M) = F(M)$. $\mathscr{T}(M)$ is the direct sum $\sum_{r,s} \mathscr{T}_s^r(M)$. $T_p$ is the value at $p \in M$ of a tensor field $T$ on $M$, and $\mathscr{T}_s^r(p)$ is the vector space of tensors of type $(r, s)$ at $p$.

(v) Let $S \in \mathscr{T}_r^s(p)$ and $T \in \mathscr{T}_s^r(p)$. Then the real number $S(T) = T(S)$ is defined, in the usual way, by contraction. It follows that if $S \in \mathscr{T}_r^s(M)$ then $S$ is a differentiable function on $T_s^rM$.

(vi) A map $D : \mathscr{T}(M) \to \mathscr{T}(M)$ is a derivation on $M$ if
  (a) $D$ is linear with respect to constant coefficients,
  (b) for all $r, s$, $D\mathscr{T}_s^r(M) \subset \mathscr{T}_s^r(M)$,
  (c) for all $C^\infty$ tensor fields $T_1$ and $T_2$ on $M$,

$$D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes DT_2 ,$$

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(d) $D$ commutes with contraction.
A derivation is determined by its action on $F(M)$ and $\mathscr{T}^1(M)$. In particular, $\mathscr{T}^1_1(M)$ may be identified with the set of derivations which map $F(M)$ to zero. The set of derivations on $M$ forms a module $\mathscr{D}M$ over $F(M)$.

(vii) The notation for covariant derivatives and curvature tensors is that of [2]. The linear connections considered on $M$ are assumed to have zero torsion.

## 3. Vector fields on $T^r_s M$

In this section we show how vector fields on $T^r_sM$ can be induced from vector fields, tensor fields of type $(r, s)$, and derivations on $M$.

We first prove a lemma which, together with its corollary, will be of use later.

**Lemma 1.** *Let $p \in M$ and $S \in \pi^{-1}(p)$. If $W$ is a vertical vector at $S$* (i.e. *tangential to $\pi^{-1}(p)$ at $S$*) *and $W(\alpha) = 0$ for all $\alpha \in \mathscr{T}^s_r(p)$ then $W = 0$.*

*Proof.* The vector space $\mathscr{T}^s_r(p)$ is dual to $\mathscr{T}^r_s(p)$ and hence $\alpha$ contains a system of coordinates on $\pi^{-1}(p)$. The result follows immediately.

**Corollary 1.** *Let $W \in \mathscr{T}^1(T^r_sM)$. If $W(\alpha)=0$ for all $\alpha \in \mathscr{T}^s_r(M)$ then $W=0$.*

*Proof.* The assumption on $W$ implies that for $\beta \in \mathscr{T}^s_{r-1}(M)$ and $f \in F(M)$,

$$0 = \frac{1}{2}W(df^2 \otimes \beta) = W((f \circ \pi)df \otimes \beta) = W(f \circ \pi)df \otimes \beta .$$

Hence $d\pi W = 0$, and so $W$ is a vertical vector field. Thus $W = 0$ by Lemma 1, the values of $W$ on the zero section of $\mathscr{T}^r_sM$ being zero by continuity.

**Proposition 1.** *Let $T \in \mathscr{T}^r_s(M)$. Then there is a unique $C^\infty$ vector field $T^v$ on $T^r_sM$ such that for $\alpha \in \mathscr{T}^s_r(M)$,*

$$T^v(\alpha) = \alpha(T) \circ \pi . \tag{1}$$

*Proof.* For $p \in M$, $\pi^{-1}(p)$ is a vector space and so $T_p$ determines a unique vertical vector field $T^v_p$ on $\pi^{-1}(p)$ such that for $\alpha \in \mathscr{T}^s_r(p)$, $T^v_p(\alpha) = \alpha(T_p)$. The cross section $T$ on $T^r_sM$ then determines a $C^\infty$ vertical vector field which satisfies (1). $T^v$ will be called the *vertical lift* of $T$.

**Corollary 2.** *Let $S \in \pi^{-1}(p)$, and let $T^v_S$ be the value of $T^v$ at $S$. Then the map $T_p \to T^v_S$ is a linear isomorphism of $\pi^{-1}(p) \to (\pi^{-1}(p))_S$, where $(\pi^{-1}(p))_S$ is the tangent space to the fibre $\pi^{-1}(p)$ at $S$.*

**Proposition 2.** *Let $D$ be a derivation on $M$. Then there is a unique vector field $\bar{D}$ on $T^r_sM$ such that for $\alpha \in \mathscr{T}^s_r(M)$*

$$\bar{D}\alpha = D\alpha . \tag{2}$$

*Proof*. Let $\{x^i\}$ $(i = 1, 2, \cdots, n)$ be a coordinate system on a neighbourhood $U$ of $p \in M$, and $\{\omega^\theta\}$ $(\theta = 1, 2, \cdots, n^{r+s})$ a basis for $\mathscr{T}_r^s(U)$. Then $\{x^i \circ \pi, \omega^\theta\}$ is a coordinate system on $\pi^{-1}(U)$. Define $\bar{D}$ on $\pi^{-1}(U)$ by

$$\bar{D}(x^i \circ \pi) = (Dx^i) \circ \pi \;, \tag{3}$$

$$\bar{D}(\omega^\theta) = D(\omega^\theta) \;. \tag{4}$$

Thus a $C^\infty$ vector field $\bar{D}$ is defined on $\pi^{-1}(U)$. Moreover, for $\alpha \in \mathscr{T}_r^s(U)$ we have $\bar{D}\alpha = D\alpha$. Hence, using Corollary 1, it follows that $\bar{D}$ is defined over $T_s^rM$ as the unique solution of (2).

**Corollary 3.** *If $f \in F(M)$ then $\bar{D}(f \circ \pi) = (Df) \circ \pi$.*

**Corollary 4.** *$\bar{D}$ is a vertical vector field if and only if $D \in \mathscr{T}_1^1(M)$.*

**Corollary 5.** *If $D_1, D_2$ are derivations on $M$, and $f_1, f_2 \in F(M)$, then $f_1D_1 + f_2D_2$ is a derivation on $M$, and*

$$\overline{f_1D_1 + f_2D_2} = (f_1 \circ \pi)\bar{D}_1 + (f_2 \circ \pi)\bar{D}_2 \;.$$

*Thus if $F(M)$ is identified with $F(M) \circ \pi = \{f \circ \pi : f \in F(M)\}$ then $D \to \bar{D}$ is a linear map of $\mathscr{D}M \to \mathscr{T}^1(T_s^rM)$.*

**Corollary 6.** *If $p \in M$ and $A \in \mathscr{T}_1^1(p)$ then for $S \in T_s^r(p)$,*

$$\bar{A}_S = -(AS)_S^v \;, \tag{5}$$

*where the suffix $S$ denotes evaluation at $S$.*

*Proof*. Let $\alpha \in \mathscr{T}_r^s(p)$. Then

$$\bar{A}_S(\alpha) = (A\alpha)(S) = -(AS)_S^v(\alpha) \;.$$

The result follows from Lemma 1.

**Corollary 7.** *Let $X \in \mathscr{T}^1(M)$ and $\mathscr{L}_X$ denote Lie derivation with respect to $X$. Then $\bar{\mathscr{L}}_X$ is a vector field on $T_s^rM$. In conformity with the notation of* [4], [8], [9], [10], *we call $\bar{\mathscr{L}}_X$ the complete lift of $X$ and write $\bar{\mathscr{L}}_X = X^c$.*

**Remark 1.** If $f \in F(M)$ then

$$\mathscr{L}_{fX} = f\mathscr{L}_X - X \otimes df \;,$$

where $X \otimes df$ is regarded as a derivation on $M$. Thus

$$(fX)^c = (f \circ \pi)X^c - \overline{X \otimes df} \;. \tag{6}$$

Now if $T_s^rM$ is the tangent bundle $T^1M$ then for $\alpha \in \mathscr{T}_1(M)$,

$$\overline{X \otimes df}(\alpha) = -\alpha(X)df \;.$$

Hence by Proposition 1,

$$\overline{X \otimes df} = -dfX^v ,$$

where $X^v$ is the vertical lift of $X$ to $T^1M$. We then have

$$(fX)^c = fX^c + dfX^v . \tag{7}$$

Equation (7) was used extensively in [8] but does not appear to extend to tensor bundles of high order. Equation (6) is perhaps a useful generalization.

**Lemma 2.** *Let $p \in M$ and $A \in \mathscr{T}_1^1(p)$. Suppose there exist non-negative integers $a$ and $b$, not both zero, such that $A\mathscr{T}_b^a(p) = 0$. Then $A = kI$ where $k$ is some real number. If $a \neq b$ then $A = 0$.*

*Proof.* We prove the lemma for the case $a > 0$. The proof for $a = 0$ and $b > 0$ is essentially the same with covariance and contravariance exchanged.

Let $S \in \mathscr{T}_b^{a-1}(p)$ be non-zero, and let $X \in \mathscr{T}^1(p)$. Then

$$AS \otimes X + S \otimes AX = 0 .$$

Choose $\omega \in \mathscr{T}_{a-1}^b(p)$ such that $\omega(S) \neq 0$. Then $(A - kI)X = 0$, where $k = -\omega(AS)/\omega(S)$. It follows immediately that $A = kI$. Then for $T \in \mathscr{T}_b^a(p)$

$$0 = AT = k(a - b)T .$$

Hence, if $a \neq b$ then $k = 0$ and $A = 0$.

**Remark 2.** $A = kI$ for some $k$ is a necessary and sufficient condition for $A\mathscr{T}_a^a(p) = 0$, $a \neq 0$.

**Corollary 8.** *Let $D \in \mathscr{D}M$ and suppose there exist non-negative integers $a$ and $b$, not both zero, such that $D\mathscr{T}_b^a(M) = 0$. Then $D = fI$, where $f \in F(M)$. If $a \neq b$ then $D = 0$.*

*Proof.* Let $h \in F(M)$ and $T \in \mathscr{T}_b^a(M)$. Then

$$(Dh)T = 0 .$$

It follows immediately that $DF(M) = 0$ and hence $D \in \mathscr{T}_1^1(M)$. Then by Lemma 2, $D = fI$ for some $f \in F(M)$, and if $a \neq b$, then $f$ is zero by Lemma 2. This completes the proof.

**Remark 3.** $D = fI$ for some $f \in F(M)$ is a necessary and sufficient condition for $D\mathscr{T}_a^a(M) = 0$, $a \neq 0$.

**Corollary 9.** *The map $D \to \bar{D}$ of $\mathscr{D}M \to \mathscr{T}^1(T_s^rM)$ is a monomorphism when $r \neq s$ and has kernel $\{fI : f \in F(M)\}$ when $r = s$.*

*Proof.* This follows from Corollaries 1, 5 and 8.

**Corollary 10.** *If $r \neq s$ then $T_s^rM$ admits a vertical vector field which vanishes only on the zero section of $T_s^rM$.*

*Proof.* The vector field $\bar{I}$ has the required properties.

**Corollary 11.** *Let $p \in M$, $A \in \mathscr{T}_1^1(p)$ and $T \in \mathscr{T}_s^r(P)$, $r \neq s$. Then $\bar{A} = T^v$ implies $A = 0$ and $T = 0$.*

*Proof*. Suppose $\bar{A} = T^v$. Then by Corollaries 2 and 6, $AS = -T$ for all $S \in \mathscr{T}^r_s(p)$. Since $A$ is linear it follows that $T = 0$ and $A\mathscr{T}^r_s(p) = 0$. Hence $A = 0$ by Lemma 2.

Suppose now that $\nabla$ is a linear connection (with zero torsion) on $M$, and let $X \in \mathscr{T}^1(M)$. Then $\nabla X \in \mathscr{T}^1_1(M)$, and hence, by Corollary 4, $\overline{\nabla X}$ is a $C^\infty$ vertical vector field on $T^r_sM$.

Another $C^\infty$ vector field $\bar{\nabla}_X$ on $T^r_sM$ is determined by the derivation $\nabla_X$. In conformity with [4] we write $\bar{\nabla}_X = X^h$, and call $X^h$ the horizontal lift of $X$. If $f \in F(M)$ then using Corollary 3,

$$X^h(f \circ \pi) = \bar{\nabla}_X(f \circ \pi) = (\nabla_X f) \circ \pi = (Xf) \circ \pi \ .$$

Hence

$$d\pi X^h = X \ . \tag{8}$$

The horizontal lift clearly satisfies

$$(fX + gY)^h = (f \circ \pi)X^h + (g \circ \pi)Y^h \ ,$$

for $f, g \in F(M)$ and $X, Y \in \mathscr{T}^1(M)$. Thus the horizontal lift is a linear map of $\mathscr{T}^1(M) \to \mathscr{T}^1(T^r_sM)$ if, as before, $F(M)$ and $F(M) \circ \pi$ are identified. Since $\bar{\nabla}_X = 0$ if and only if $X = 0$, the horizontal lift is a monomorphism, and so determines a horizontal subspace $H_S$ of dimension $n(= \dim M)$ at each point $S \in T^r_sM$. Then $C^\infty$ distribution $H$ on $T^r_sM$ so obtained is usually called the horizontal distribution determined by the connection $\nabla$.

If $S \in T^r_sM$ then the tangent space $(T^r_sM)_S$ is the direct sum $V_S + H_S$, where $V_S$ is the subspace of vertical vectors at $S$. Thus, if $W \in (T^r_sM)_S$ then

$$W = h(W) + v(W) \ ,$$

where $h$ and $v$ are the projections onto the horizontal and vertical subspaces at $S$. Clearly $X^h = h(X^h)$ and $T^v = v(T^v)$ for any vector $X$ and tensor $T$ of type $(r, s)$ at $\pi(S)$.

## 4. Lie brackets

We now determine, for later use, the Lie brackets of some particular types of vector fields on $T^r_sM$. These results generalize some of those already obtained for tangent bundles and cotangent bundles [1], [4], [7], [8], [9], [10].

**Lemma 3.** *Let $T_1, T_2 \in \mathscr{T}^r_s(M)$ and $X, X_1, X_2 \in \mathscr{T}^1(M)$, and let $D, D_1, D_2$, $A$ be derivations on $M$, where $A \in \mathscr{T}^1_1(M)$. Let $R$ denote the curvature tensor field of the connection $\nabla$. Then*

$$[T_1^v, T_2^v] = 0 \ , \tag{9}$$

$$(10) \qquad [\bar{D}_1, \bar{D}_2] = \overline{[D_1, D_2]} ,$$

$$(11) \qquad [\bar{D}, T^v] = (DT)^v ,$$

$$(12) \qquad [X^h, T^v] = (\nabla_X T)^v ,$$

$$(13) \qquad [X_1^h, X_2^h] = \overline{R(X_1, X_2)} + [X_1, X_2]^h ,$$

$$(14) \qquad [X^h, \bar{A}] = \overline{\nabla_X A} ,$$

$$(15) \qquad [X_1^c, X_2^c] = [X_1, X_2]^c .$$

*Proof.* Several equations can be proved by application of Corollary 1.

If $p \in M$ then $\pi^{-1}(p)$ is a vector space, and has the structure of an abelian Lie group. If $S \in \mathscr{T}_s^r(M)$ then $S^v$ is an invariant vector field on $\pi^{-1}(p)$ and equation (9) follows immediately.

We have, from Proposition 2,

$$[\bar{D}_1, \bar{D}_2]\alpha = (\bar{D}_1\bar{D}_2 - \bar{D}_2\bar{D}_1)\alpha = [D_1, D_2]\alpha .$$

Since $[D_1, D_2]$ is a derivation on $M$, from Proposition 2 we have

$$[D_1, D_2]\alpha = \overline{[D_1, D_2]}\alpha ,$$

and hence equation (10).

$$[\bar{D}, T^v]\alpha = (D(\alpha(T)) - (D\alpha)(T)) \circ \pi = (\alpha(DT)) \circ \pi = (DT)^v(\alpha) ,$$

which gives equation (11). Since $X^h = \bar{\nabla}_X$, equation (12) is a special case of (11).

Since $R(X_1, X_2) \in \mathscr{T}_1^1(M)$, we have

$$[X_1^h, X_2^h] = [\bar{\nabla}_{X_1}, \bar{\nabla}_{X_2}] = \overline{[\nabla_{X_1}, \nabla_{X_2}]} = \overline{R(X_1, X_2)} + \bar{\nabla}_{[X_1, X_2]} ,$$

from which follows immediately equation (13).

$$[X^h, \bar{A}]\alpha = \nabla_X(A\alpha) - A(\nabla_X\alpha) = (\nabla_X A)\alpha = (\overline{\nabla_X A})\alpha ,$$

which gives equation (14). Since $X^c = \bar{\mathscr{L}}_X$, equation (15) is a special case of (10).

## 5. Almost complex structures

We now consider the main problem, that is, to determine a class of tensor bundles which admit almost complex structures. For this purpose it is sufficient to consider contravariant tensor bundles since a Riemannian metric tensor field induces a fibre preserving diffeomorphism of $T_s^r M \to T^{r+s} M$. Also the tangent bundle $T^1M$ of a Riemannian space always admits an almost complex structure [1], [5]. Hence we shall restrict attention to $T^rM$, $r > 1$.

**Lemma 4.** *Let $\nabla$ and $g$ be, respectively, a symmetric connection and a Riemannian metric tensor field on $M$, and $E \in \mathscr{T}^{r-1}(M)$ be nowhere zero on $M$. Then $T^rM$ admits three distributions which are mutually orthogonal with respect to a Riemannian metric tensor field $\bar{g}$ induced on $T^rM$ by $\nabla$ and $g$.*

*Proof.* For each $p \in M$ a scalar product $<,>$ is defined on the vector space $\pi^{-1}(p)$ by $< T_1, T_2 > = t_1(T_2)$, where, for any tensor $T$ with components $T^{i_1 i_2 \cdots i_r}$, $t$ is the covariant tensor associated to $T$ by $g$. Thus $t$ has components

$$t_{i_1 i_2 \dots i_r} = T^{j_1 j_2 \cdots j_r} g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_r j_r} ,$$

where each repeated suffix indicates summation over its range. If $S \in T^rM$, then a scalar product, denoted by the same symbol $<,>$, is defined on the vector space $(T^rM)_S$ by the three equations

$$< T_1^v, T_2^v > = < T_1, T_2 > \circ \pi , \tag{16}$$

$$< T^v, X^h > = 0 , \tag{17}$$

$$< X_1^h, X_2^h > = < X_1, X_2 > \circ \pi , \tag{18}$$

where $X^h$ is the horizontal lift of $X$ with respect to $\nabla$. These equations are easily seen to determine $\bar{g}$ on $T^rM$ with respect to which the horizontal distribution $H$, induced by $\nabla$, is orthogonal to the fibres of $T^rM$ [3].

We now make use of $E$. For $X \in \mathscr{T}^1(M)$, define the vertical lift $X_E^v$ of $X$ with respect to $E$ by

$$X_E^v = (E \otimes X)^v .$$

The map $X \to X_E^v$ is then a monomorphism of $\mathscr{T}^1(M) \to \mathscr{T}^1(T^rM)$. Hence an $n$-dimensional $C^\infty$ vertical distribution $V^E$ is defined on $T^rM$. Let $V^\perp$ be the distribution on $T^rM$ which is orthogonal to $H$ and $V^E$. Then $H$, $V^E$ and $V^\perp$ are the required distributions and the proof is complete.

We now give an alternative characterization of $V^\perp$.

**Lemma 5.** *Let $p \in M, S \in \pi^{-1}(p)$, and $\mathscr{T}_E^r(p)$ be the subspace of $\mathscr{T}^r(p)$ defined by*

$$\mathscr{T}_E^r(p) = \{T : < T, E \otimes X > = 0 \text{ for all } X \in \mathscr{T}^1(p)\} .$$

*Then $V_S^\perp = (\mathscr{T}_E^r(p))_S^v$.*

*Let $E^\perp(p)$ be the subspace of $\mathscr{T}^{r-1}(p)$ defined by*

$$E^\perp(p) = \{T : < T, E > = 0\} .$$

*Then $\mathscr{T}_E^r(p) = E^\perp(p) \otimes \mathscr{T}^1(p)$.*

*Proof.* The first part of the lemma follows from the fact that the vertical lift preserves scalar products. To prove the second part it is sufficient to note that $E^{\perp}(p)\otimes\mathscr{T}^1(p)\subset\mathscr{T}^r_E(p)$, and

$$\dim(E^{\perp}(p)\otimes\mathscr{T}^1(p)) = n(n^{r-1}-1) = n^r - n = \dim\mathscr{T}^r_E(p)\ .$$

**Theorem.** *If $M$ admits an almost complex structure and a nowhere zero tensor field $E\in\mathscr{T}^{r-1}(M)$, then $T^rM$ admits an almost complex structure.*

*Proof.* Let $F$ be an almost complex structure on $M$. We define a $C^\infty$ tensor field $J$ of type (1,1) on $T^rM$ by its action on the distributions $H$, $V^E$ and $V^{\perp}$. Thus for $X\in\mathscr{T}^1(M)$ and $T\in\mathscr{T}^r(M)$ define $J$ by

$$J(X^h) = X^v_E,\ J(X^v_E) = -X^h,\ J(T^v) = \tilde{T}^v\ , \tag{19}$$

where $\tilde{T}$ is obtained by contracting $T\otimes F$, and has components $T^{i_1i_2\cdots i_{r-1}l}F_l^{i_r}$, where $T^{i_1i_2\cdots i_r}$ and $F_j^i$ are local components of $T$ and $F$ respectively. The restrictions of $J$ to $H+V^E$ and $V^{\perp}$ are endomorphisms, and hence $J$ is a tensor field on $T^rM$. It is easily seen that $J$ is $C^\infty$ and $J^2=-I$, $I$ being the unit tensor. Hence $J$ is an almost complex structure on $T^rM$.

**Corollary 12.** *Suppose a Riemannian manifold $M$ admits an almost complex structure. Then $T^rM$ admits an almost complex structure if* (i) *$r$ is odd or* (ii) *$r$ is even and $M$ admits a nowhere zero vector field.*

*Proof.* (i) For $r=2s+1$ choose $E=(\otimes g^{-1})^s$, where $g^{-1}$ is the inverse of a metric tensor field $g$ on $M$, and $(\otimes g^{-1})^s$ is the tensor product of $g^{-1}$ with itself $s$ times.

(ii) For $r=2s$, $s>1$, choose $E=(\otimes g^{-1})^{s-1}\otimes X$, where $M$ is assumed to admit a nowhere zero vector field $X$. For $r=2$ choose $E=X$.

## 6. Integrability of the almost complex structure $J$

We now establish necessary and sufficient conditions for the integrability of $J$.

Let $e$ be the covariant tensor field of order $r-1$ associated to $E$ by $g$; thus, with respect to local coordinates, $e$ has components $e_{i_1i_2\dots i_{r-1}}$ given by

$$e_{i_1i_2\dots i_{r-1}} = g_{i_1j_1}g_{i_2j_2}\dots g_{i_{r-1}j_{r-1}}E^{j_1j_2\cdots j_{r-1}}\ .$$

**Proposition 3.** *Suppose $M$ admits an almost complex structure $F$ and a nowhere zero tensor field $E\in\mathscr{T}^{r-1}(M)$. Then the induced almost complex structure $J$ is integrable if and only if, for $X, Y\in\mathscr{T}^1(M)$,*

$$R(X,Y)=0,\quad \nabla_XE=0,\quad \nabla_XF=0,\quad \nabla_X\frac{e}{<E,E>}=0\ .$$

*Proof.* Let $N$ be the Nijenhuis 2-form on $T^r M$ with values in $\mathscr{T}^1(T^r M)$, defined by

$$N(W_1, W_2) = [W_1, W_2] + J[JW_1, W_2] + J[W_1, JW_2] - [JW_1, JW_2]$$

for $W_1, W_2 \in \mathscr{T}^1(T^r M)$. Then $J$ is integrable if and only if $N = 0$.

Suppose $N = 0$. Then for $X, Y \in \mathscr{T}^1(M)$, $N(X_E^v, Y_E^v) = 0$. Hence, putting $W_1 = X_E^v$, $W_2 = Y_E^v$ we have, from (9), (12), (13), and the definition of $J$,

$$\begin{aligned} \overline{R(X, Y)} &= J(\nabla_Y(E \otimes X))^v - J(\nabla_X(E \otimes Y))^v - [X, Y]^h \\ &= J((\nabla_Y E) \otimes X)^v - J((\nabla_X E) \otimes Y)^v - (\nabla_Y X)^h \\ &\qquad + (\nabla_X Y)^h - [X, Y]^h \\ &= J((\nabla_Y E) \otimes X - (\nabla_X E) \otimes Y)^v \end{aligned} \tag{20}$$

since $\nabla$ has zero torsion. Now since $E \otimes \mathscr{T}^1(M)$ is a subspace of $\mathscr{T}^r(M)$ there is a unique $T \in \mathscr{T}^r(M)$ orthogonal to this subspace and a unique $Z \in \mathscr{T}^1(M)$ such that

$$(\nabla_Y E) \otimes X - (\nabla_X E) \otimes Y = T + E \otimes Z .$$

Then from (19) and (20)

$$\overline{R(X, Y)} = \tilde{T}^v - Z^h .$$

Since $\overline{R(X, Y)}$ is vertical, $Z^h = 0$ and hence $Z = 0$. It follows from Corollary 11 that

$$R(X, Y) = 0 , \tag{21}$$

$$T = 0 . \tag{22}$$

We thus have for all $X, Y \in \mathscr{T}^1(M)$,

$$(\nabla_X E) \otimes Y = (\nabla_Y E) \otimes X .$$

Since $M$ is assumed to admit an almost complex structure, dim $M \geq 2$. Hence by choosing $X, Y$ to be linearly independent it follows that

$$\nabla_X E = 0 . \tag{23}$$

We next consider the case $N(X_E^v, T^v) = 0$, where $X_E^v \in V^E$ and $T^v \in V^\perp$. Then from (9), (12) and the definition of $J$ we have

$$J(\nabla_X T)^v = (\nabla_X \tilde{T})^v . \tag{24}$$

It follows that $(\nabla_X T)^v \in V^\perp$. Choose $T = S \otimes Y$ where $S \in \mathscr{T}^{r-1}(M)$, $Y \in \mathscr{T}^1(M)$

and $< S, E > = 0$ (since $M$ is paracompact such an $S$ exists and can be chosen to be non-zero in a neighbourhood of a point). Then by Lemma 5, $T^v \in V^\perp$ and (24) imply that

$$(\nabla_X S) \otimes FY + S \otimes F\nabla_X Y = (\nabla_X S) \otimes FY + S \otimes \nabla_X(FY) .$$

Hence

$$S \otimes (\nabla_X F)Y = 0 ,$$

and it follows immediately that

$$\nabla_X F = 0 . \tag{25}$$

Finally, from Lemma 5 the condition $(\nabla_X T)^v \in V^\perp$ implies that

$$0 = e(\nabla_X S) = -(\nabla_X e)S . \tag{26}$$

But $S$ is any tensor field which satisfies $< S, E > = 0$. Hence we deduce that

$$\nabla_X e = \alpha(X)e , \tag{27}$$

where $\alpha \in \mathscr{T}_1(M)$. Then $\alpha$ is determined by

$$\alpha(X) = \frac{(\nabla_X e)(E)}{e(E)} = \frac{X(e(E))}{e(E)} = \frac{X < E, E >}{< E, E >} .$$

Thus

$$\alpha = d \log e(E) = d \log < E, E > . \tag{28}$$

(If $\nabla$ is the Riemannian connection associated with $g$ then (23) implies (27) and $\alpha = 0$.) Hence, from (27) and (28), the tensor field $\dfrac{e}{< E, E >}$ has zero covariant derivative. This proves the necessity of the conditions in Proposition 3.

To prove the sufficiency we note that

$$N(X_E^v, Y_E^v) = N(Y^h, X^h) = JN(Y_E^v, X^h) ,$$
$$N(X_E^v, T^v) = JN(T^v, X^h), \quad N(T_1^v, T_2^v) = 0 .$$

Thus $N = 0$ if $N(X_E^v, Y_E^v) = N(X_E^v, T^v) = 0$. Suppose $\nabla_X E = 0$ and $R(X, Y) = 0$ for all $X, Y \in \mathscr{T}^1(M)$. Then

$$\begin{aligned} N(X_E^v, Y_E^v) &= -J[X^h, Y_E^v] - J[X_E^v, Y^h] - [X^h, Y^h] \\ &= (\nabla_X Y)^h - (\nabla_Y X)^h - [X, Y]^h = 0 . \end{aligned}$$

Suppose $\nabla_X \dfrac{e}{<E, E>} = 0$. Then (27) follows and hence if $T^v \in V^\perp$ then $(\nabla_X T)^v \in V^\perp$. If we next assume $\nabla_X F = 0$ then we have

$$N(X_E^v, T^v) = (\nabla_X \tilde{T})^v - J(\nabla_X T)^v = 0 ,$$

which proves the sufficiency.

## 7. Kählerian structure on $T^r M$

We now determine necessary and sufficient conditions for the metric $\bar{g}$ on $T^r M$, defined in §5, to be Kählerian with respect to $J$.

**Proposition 4.** *$\bar{g}$ is Hermitian with respect to $J$ if and only if $<E, E> = 1$ and $g$ is Hermitian with respect to $F$.*

*Proof.* Suppose $\bar{g}$ is Hermitian with respect to $J$. Then for $X, Y \in \mathscr{T}^1(M)$,

$$\begin{aligned} <X, Y> \circ \pi &= <X^h, Y^h> = <JX_E^v, JY_E^v> = <X_E^v, Y_E^v> \\ &= <E \otimes X, E \otimes Y> \circ \pi = <E, E><X, Y> \circ \pi . \end{aligned}$$

Hence $<E, E> = 1$. Now let $p \in M$ and let $S \in \mathscr{T}^{r-1}(p)$ be non-zero such that $<S, E> = 0$. Then by Lemma 5 and the definition of $J$ we have, for $X, Y \in \mathscr{T}^1(p)$,

$$\begin{aligned} <S, S><X, Y> \circ \pi &= <S \otimes X, S \otimes Y> \circ \pi \\ &= <(S \otimes X)^v, (S \otimes Y)^v> = <J(S \otimes X)^v, J(S \otimes Y)^v> \\ &= <S \otimes FX, S \otimes FY> \circ \pi = <S, S><FX, FY> \circ \pi . \end{aligned}$$

Thus at $p$, $<X, Y> = <FX, FY>$. Since $p$ is arbitrary, $g$ is Hermitian with respect to $F$. The sufficiency of the above conditions is easily proved by the same method.

**Proposition 5.** *Suppose $\bar{g}$ is Hermitian with respect to $J$. Then $\bar{g}$ is Kählerian with respect to $J$ if and only if $\nabla$ is the Riemannian connection associated with $g$, $R = 0$, $\nabla E = 0$ and $\nabla F = 0$.*

*Proof.* Let $\alpha$ be the field of 2-forms on $T^r M$ defined for all $W_1, W_2 \in \mathscr{T}^1(T^r M)$ by $\alpha(W_1, W_2) = <W_1, JW_2>$. Then $\bar{g}$ is Kählerian with respect to $J$ if and only if $\alpha$ is closed and $J$ is integrable [6, Chapter VII]. As usual it is sufficient to consider the action of $\alpha$ and $d\alpha$ on the three distributions $H$, $V^E$ and $V^\perp$ on $T^r M$. Then for $X, Y \in \mathscr{T}^1(M)$ and $T_1^v, T_2^v \in V^\perp$ we have

$$\alpha(X_E^v, Y_E^v) = \alpha(X^h, Y^h) = \alpha(T_1^v, X_E^v) = \alpha(T_1^v, X^h) = 0 ,$$

$$\alpha(X_E^v, Y^h) = <E \otimes X, E \otimes Y> \circ \pi = <X, Y> \circ \pi , \tag{29}$$

$$\alpha(T_1^v, T_2^v) = <T_1, \tilde{T}_2> \circ \pi .$$

Suppose $\bar{g}$ is Kählerian with respect to $J$. Then by Propositions 3 and 4, $R=0$, $\nabla_X E=0$, and $\nabla_X e=0$, for all $X \in \mathscr{T}^1(M)$. Let $p \in M$, $X \in \mathscr{T}^1(p)$, and choose $T \in \mathscr{T}^{r-1}(M)$ such that $<T, E> = 0$ and $<T, T> = 1$ on some neighbourhood $U$ of $p$. Since $R=0$ parallel vector fields $Y$ and $Z$ exist on $U$ with arbitrary initial values at $p$. Then using (9), (12) and Lemma 5 we have, on $\pi^{-1}(p)$,

$$
\begin{aligned}
0 &= d\alpha((T \otimes Y)^v, (T \otimes X)^v, X^h) \\
&= X < T \otimes Y, T \otimes FX > + < T \otimes FY, \nabla_X(T \otimes Z) > \\
&\qquad - < \nabla_X(T \otimes Y), T \otimes FZ > \\
&= X < Y, FZ > + 2 < T, \nabla_X T >< FY, Z > \\
&\qquad + <FY, \nabla_X Z> - < \nabla_X Y, FZ > \\
&= (\nabla_X g)(Y, FZ) - 2 < T, \nabla_X T><Y, FZ> .
\end{aligned} \tag{30}
$$

Since $F$ is non-singular it follows that

$$\nabla_X g = \alpha(X) g ,$$

for some $\alpha \in \mathscr{T}_1(p)$. Then since $\nabla_X E = 0$ and $\nabla_X e = 0$ it follows easily that for all $X \in \mathscr{T}^1(p)$,

$$0 = \nabla_X e = (r-1)\alpha(X) e .$$

The tensor $e$ is non-zero and so $\alpha = 0$. Thus $\nabla g = 0$ at $p$ and hence on $M$ since $p$ is arbitrary. It follows that $\nabla$, having no torsion, is the Riemannian connection associated with $g$.

We now prove the sufficiency of the above conditions by showing that the 2-form $\alpha$ is exact. Let $X \in \mathscr{T}^1(M)$, and $T^v \in V^\perp$. Define a 1-form $\beta$ on $T^r M$ as follows: at each point $S \in T^r M$,

$$\beta(X^h) = <S, E \otimes X>, \quad \beta(X^v_E) = 0, \quad \beta(T^v) = \tfrac{1}{2} <S, \bar{T}> .$$

Then using (29) it follows after some calculation that $\alpha = d\beta$. Hence $d\alpha = 0$, and this together with Proposition 3 proves the sufficiency.

## 8. Integrability of $H + V^E$ and $H + V^\perp$

**Proposition 6.** *$H + V^E$ is integrable if and only if $R=0$ and for $X \in \mathscr{T}^1(M)$, $\nabla_X E = \alpha(X) E$, where $\alpha(X) = \dfrac{<E, \nabla_X E>}{<E, E>}$.*

*Proof*. It follows from (12) and (13) that $H + V^E$ is an integrable distribution if and only if for $X_1, X_2 \in \mathscr{T}^1(M)$,

$$(\nabla_{X_1}(E \otimes X_2))^v \in V^E , \tag{31}$$

(32) $$\overline{R(X_1, X_2)} \in V^E .$$

Let $Y_1$ and $Y_2$ be orthogonal vectors at $p \in M$, and let $<T, E> = 0$ at $p$. Then from (16), (32) and Corollary 6,

$$0 = <R(X_1, X_2)(T \otimes Y_1), T \otimes Y_2> = <T, T><R(X_1, X_2)Y_1, Y_2> .$$

Hence $R(X_1, X_2)Y_1 = cY_1$ where $c$ is some real number which depends on $X_1$ and $X_2$. Since $Y_1$ is arbitrary it follows that $R(X_1, X_2) = cI$ at $p$. Then at any point $S \in \pi^{-1}(p)$ we have $\overline{R(X_1, X_2)} = -crS^v$, and by choosing $S^v \in V^\perp$ it follows that $\overline{R(X_1, X_2)} = 0$ at $S$; hence $c = 0$. Since $p$, $X_1$ and $X_2$ are arbitrary we have $R = 0$ on $M$.

Using (30) and Lemma 5 we obtain $\nabla_X E = \alpha(X)E$ and $\alpha$ is then uniquely determined by this equation.

The proof of the sufficiency is immediate.

**Proposition 7.** *$H + V^\perp$ is integrable if and only if $R=0$ and for $X \in \mathscr{T}^1(M)$, $\nabla_X e = \alpha(X)e$, where* $\alpha = \dfrac{<e, \nabla_X e>}{<e, e>}$.

*Proof.* The proof is similar to that of Proposition 6 and we shall use the same notation. It follows from (12), (13) and Lemma 5 that $H + V^\perp$ is an integrable distribution if and only if for $S^v \in V^\perp$,

(33) $$(\nabla_{X_1}(S \otimes X_2))^v \in V^\perp ,$$

(34) $$\overline{R(X_1, X_2)} \in V^\perp .$$

then from (16), (34) and Corollary 6,

$$0 = <R(X_1, X_2)(E \otimes Y_1), E \otimes Y_2> = <E, E><R(X_1, X_2)Y_1, Y_2> .$$

Hence, as before, $R = 0$.

From (33) we obtain

$$0 = <\nabla_{X_1}S, E><X_2, Y>$$

for $Y \in \mathscr{T}^1(p)$. Hence

$$0 = <\nabla_{X_1}S, E> = e(\nabla_{X_1}S) = -(\nabla_{X_1}e)S .$$

It follows that $\nabla_{X_1}e = \alpha(X_1)e$ at $p$. Since $p$ and $X_1$ are arbitrary we obtain $\nabla_{X_1}e = \alpha(X_1)e$ on $M$, and $\alpha$ is then uniquely determined.

The proof of the sufficiency is immediate.

## References

[1] P. Dombrowski, *On the goemetry of tangent bundles,* J. Reine Angew. Math. **210** (1962) 73–88.
[2] S. Helgason, *Differential geometry and symmetric spaces,* Academic Press, New York, 1962.
[3] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds,* Tôhoku Math. J. **10** (1958) 338–354.
[4] A. J. Ledger & K. Yano, *The tangent bundle of a locally symmetric space,* J. London Math. Soc. **40** (1965) 487–492.
[5] S. Tachibana & M. Okumura, *On the almost complex structure of tangent bundles of Riemannian spaces,* Tôhoku Math. J. **14** (1962) 158–161.
[6] K. Yano, *Differential geometry on complex and almost complex spaces,* Pergamon, New York, 1965.
[7] K. Yano & S. Ishihara, *Almost complex structures induced in tangent bundles,* Kōdai Math. Sem. Rep. **19** (1967) 1–27.
[8] K. Yano & S. Kobayashi, *Prolongations of tensor fields and connections to tangent bundles* I, J. Math. Soc. Japan **18** (1966) 194–210.
[9] K. Yano & A. J. Ledger, *Linear connections on tangent bundles,* J. London Math. Soc. **39** (1964) 495–500.
[10] K. Yano & E. M. Patterson, *Vertical and complete lifts from a manifold to its cotangent bundle,* J. Math. Soc. Japan **19** (1967) 91–113.

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